

Existence of Extremal Solution of FOFD Equation

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Abstract:

In this paper the existence theorem for the first order functional differential equations in Banach algebras is proved under the mixed generalized Lipschitz and Caratheodory conditions.

Key Words: Measure Integro-differential equation, Existence theorem, Extremal solutions.

1. Introduction

The existence of extremal solutions is also proved under certain monotonicity conditions. As a generalization of ordinary integro-differential equations, there is a series of papers dealing with the abstract measure integro-differential equations in which ordinary derivative is replaced by the derivative of set functions, namely, the Radon-Nokodym derivative of a measure with respect to another measure. See Dhage [3,4], and the references therein. The above mentioned papers also include some already known abstract measure differential equations those considered in P. C. Das and Sharma [2], Shendge and Joshi studied as special cases. The origin of the quadratic integral equations appears in the works of Chandrasekhar's H-equation in radioactive heat transfer, but the study of nonlinear integral equations via operator theoretic techniques seems to have been started by Dhage in the year 1988. Similarly, the study of nonlinear quadratic differential equations is relatively new and initiated by Dhage and O'Regan in the year 2000.

1.1 Statement of Problem

Let $\mathbf{R}$ denote the real line and let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbf{R}$. Let $J = I_0 \cup I$, then J is a closed and bounded interval in $\mathbf{R}$. Let C denote the Banach space of all continuous real-valued functions ϕ on I_0 with the supremum norm $\|\cdot\|_C$ defined by

$$\|\phi\|_C = \sup_{t \in I_0} |\phi(t)|$$

Clearly C is a Banach algebra with this norm. Consider the first order functional differential equation (in short FDE)

$$\left. \begin{aligned} \left(\frac{x(t)}{f(t, x(t))} \right)' &= g(t, x_t) \quad \text{a.e. } t \in I \\ x(t) &= \phi(t), \quad t \in I_0 \end{aligned} \right\} \quad (1.1)$$

where $f : I \times \mathbf{R} \rightarrow \mathbf{R} - \{0\}$, $g : I \times C \rightarrow \mathbf{R}$ and the function $x_t : I_0 \rightarrow \mathbf{C}$ is defined by $x_t(\theta) := x(t + \theta)$ for all $\theta \in I_0$.

By a solution of FDE (1.1) we mean a function $x \in C(J, \mathbf{R}) \cap AC(I, \mathbf{R}) \cap C(I_0, \mathbf{R})$ that satisfies the equations in (1.1), where $AC(I, \mathbf{R})$ is the space of all absolutely continuous real valued functions on J .

The functional differential equations have been the most active area of research since long time. see Hale, Henderson and the references therein. But the study of functional differential equations in Banach algebra is very rare in the literature. Very recently the study along this line has been initiated via fixed point

theorems. see Dhage and Regan [5] and Dhage [6] and the references therein. The FDE (1.1) is new to the literature and the study of this problem will definitely contribute immensely to the area of functional differential equations. The fixed-point theorem of Dhage [4] we will be using are given in the following section.

2. Auxiliary Results

Let X be a Banach algebra with norm $\|\cdot\|$. A mapping $A: X \rightarrow X$ is called D-Lipschitzian if there exists a continuous nondecreasing function $\psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \leq \psi(\|x - y\|) \quad (2.1)$$

For all $x, y \in X$ with $\psi(0) = 0$. In the special case when $\psi(r) = \alpha r$ ($\alpha > 0$), A is called a Lipschitzian with a Lipschitz constant α . In particular, if $\alpha < 1$, A is called a contraction with a contraction constant α . Further, if $\psi(r) < r$ for all $r > 0$, then A is called a nonlinear contraction on X . Sometimes we call the function ψ a **D-function** for convenience.

An operator $T: X \rightarrow X$ is called compact if $\overline{T(X)}$ is a compact subset of X . Similarly, $T: X \rightarrow X$ is called totally bounded if T maps a bounded subset of X into the relatively compact subset of X . Finally, $T: X \rightarrow X$ is called completely continuous operator if it is continuous and totally bounded operator on X . It is clear that every compact operator is totally bounded, but the converse may not be true. The nonlinear alternative of Schaefer type recently proved by Dhage [6] is embodied in the following theorem.

Theorem 2.1 (Dhage[6]) Let X be a Banach algebra and let $A, B: X \rightarrow X$ be two operators satisfying

- (a) A is a D-Lipschitzian with a D-function ψ ,
- (b) B is compact and continuous, and

(c) $M\psi(r) < r$ whenever $r > 0$, where $M = \|B(X)\| = \sup\{\|B(x)\| : x \in X\}$

Then either

- (i) The equation $\lambda A\left(\frac{x}{\lambda}\right) Bx = x$ has a solution for $\lambda = 1$, or
- (ii) The set $\varepsilon = \{u \in X : \lambda A(x/\lambda) Bx = u, 0 < \lambda < 1\}$ is unbounded.

It is known that theorem 2.1 is useful for proving the existence theorems for the integral equations of mixed type. See [5] and the references therein. The method is commonly known as priori bound method for the nonlinear equations. See, for example, Dugunji and Granas, Zeidler and the references therein.

An interesting corollary to Theorem 2.1 in its applicable form is

Corollary 2.1 Let X be a Banach algebra and let $A, B: X \rightarrow X$ be two operators satisfying

- (a) A is Lipschitzian with a Lipschitz constant α ,
- (b) B is compact and continuous, and

(c) $\alpha M < 1$, where $M = \|B(X)\| := \sup\{\|B(x)\| : x \in X\}$.

Then either

- (i) The equation $\lambda A\left(\frac{x}{\lambda}\right) Bx = x$ has a solution for $\lambda = 1$, or
- (ii) The set $\varepsilon = \{u \in X : \lambda A(x/\lambda) Bx = u, 0 < \lambda < 1\}$ is unbounded.

3. Existence Theory

Let $M(J, \mathbb{R})$ and $B(J, \mathbb{R})$ respectively denote the spaces of measurable and bounded real-valued

functions on J . We shall seek the existence of a solution of FDE (1.1) in the space $C(J, R)$, of all absolutely continuous real – valued functions on J . Define a norm $\|\cdot\|$ in $C(J, R)$ by

$$\|x\| = \sup_{t \in J} |x(t)|.$$

Clearly $C(J, R)$ becomes a Banach algebra with this norm. Note that $C(J, R) \subset AC(J, R)$. We need the following definition in the sequel.

Definition 3.1 A mapping $\beta: I \times C \rightarrow R$ is said to satisfy the condition of $L^1_{x^-}$ **Caratheodory** or simply said to be $L^1_{x^-}$ **Caratheodory** if

- (i) $t \rightarrow \beta(t, x)$ is measurable for each $x \in C$
- (ii) $x \rightarrow \beta(t, x)$ is a continuous almost everywhere for $t \in I$, and
- (iii) there exists a function $h \in L^1(I, R)$ such that

$$|\beta(t, x)| \leq h(t), \text{ a.e. } t \in I$$

For all $x \in R$.

For convenience, the function h is referred to as a **bound function** of β . We will need the following hypothesis in the sequel.

(H₁) : The function $f: I \times R \rightarrow R$ is continuous and there exists a function $k \in B(I, R)$ such that $k(t) > 0$, a.e. $t \in I$ and

$$|f(t, x) - f(t, y)| \leq k(t)|x - y|, \text{ a.e. } t \in I$$

For all $x, y \in R$.

(H₂) : $f(0, \phi(0)) = 1$.

(H₃) : The function $g(t, x)$ is $L^1_{x^-}$ **Caratheodory** with the bound function h .

(H₄) : There exists a continuous and nondecreasing function $\Omega: [0, \infty) \rightarrow (0, \infty)$ and a function $\gamma \in L^1(I, R)$ such that $\gamma(t) > 0$, a.e. $t \in I$ and

$$|g(t, x)| \leq \gamma(t)\Omega(\|x\|_C), \text{ a.e. } t \in I,$$

For all $x \in C$.

Theorem 3.1 Assume that the hypothesis (H₁)-(H₄) hold. Suppose that

$$\int_{C_1}^{\infty} \frac{ds}{\Omega(s)} > C_2 \|\gamma\|_{L^1} \quad (3.1)$$

Where,

$$C_1 = \frac{F \|\phi\|_C}{1 - \|k\|(\|\phi\|_C + \|h\|_{L^1})}, \quad C_2 = \frac{1}{1 - \|k\|(\|\phi\|_C + \|h\|_{L^1})}, \quad \|k\|(\|\phi\|_C + \|h\|_{L^1}) < 1$$

$$F = \max_{t \in J} |f(t, 0)|, \text{ and } \|k\| = \max_{t \in J} |k(t)|.$$

Then the FDE(1.1) has a solution on J .

Proof : Now the FDE(1.1) is equivalent to the functional integral equation (in short FIE)

$$x(t) = [f(t, x(t))] \left(\phi(0) + \int_0^t g(s, x_s) ds \right), \text{ if } t \in I \quad (3.2)$$

and

$$x(t) = \phi(t), \quad \text{if } t \in I_0 \quad (3.3)$$

Define the two mappings A and B on $C(J, R)$ by

$$Ax(t) = \begin{cases} f(t, x(t)), & \text{if } t \in I, \\ 1 & \text{if } t \in I_0. \end{cases} \quad (3.4)$$

And

$$Bx(t) = \begin{cases} \phi(0) + \int_0^t g(s, x_s) ds, & \text{if } t \in I, \\ \phi(t) & \text{if } t \in I_0. \end{cases} \quad (3.5)$$

Obliviously A and B define the operators $A, B : C(J, R) \rightarrow C(J, R)$. then the FDE(1.1) is equivalent to the operator equation

$$x(t) = Ax(t)Bx(t), \quad t \in J \quad (3.6)$$

We shall show that A and B satisfy all the hypothesis of corollary 2.1

We first show that A is Lipschizian on $C(J, R)$. Let $x, y \in C(J, R)$. then by (H_1) ,

$$\begin{aligned} \|Ax(t) - Ay(t)\| &\leq |f(t, x(t)) - f(t, y(t))| \\ &\leq k(t)|x(t) - y(t)| \\ &\leq k(t)\|x - y\| \end{aligned}$$

For all $t \in J$. Taking the supremum over t we obtain

$$\|Ax - Ay\| \leq \|k\| \|x - y\|.$$

For all $x, y \in C(J, R)$. So A is a Lipschitzian on $C(J, R)$ with a Lipschitz constant $\|k\|$.

Next we show that B is completely continuous on $C(J, R)$. Using the standard arguments as in Granas et al., it is shown that B is a continous operator on $C(J, R)$.

Let S be a bounded set in $C(J, R)$. We shall show that $B(C(J, R))$ is a uniformly bounded and equicontinuous set in $C(J, R)$. Since $g(t, x_t)$ is $L^1_{x^*}$ -Caratheodory, we have

$$\begin{aligned} |Bx(t)| &\leq \|\phi\|_C + \int_0^t |g(s, x_s)| ds \\ &\leq \|\phi\|_C + \int_0^t h(s) ds \\ &\leq \|\phi\|_C + \|h\|_{L^1} \end{aligned}$$

Taking the supremum over t, we obtain $\|Bx\| \leq M$ for all $x \in S$, where

$$M = \|\phi\|_C + \|h\|_{L^1}.$$

This shows that $B(C(J, R))$ is uniformly bounded set in $C(J, R)$.

Now we show that $B(C(J, R))$ is an equicontinuous set. $t, \tau \in I$. Then for any $x \in C(J, R)$ we have by (3.4),

$$\begin{aligned} |Bx(t) - Bx(\tau)| &\leq \left| \int_0^t g(s, x_s) ds - \int_0^\tau g(s, x_s) ds \right| \\ &\leq \left| \int_\tau^t g(s, x_s) ds \right| \\ &\leq \left| \int_\tau^t h(s) ds \right| \end{aligned}$$

$$\leq |p(t) - p(\tau)|$$

$$\text{where } p(t) = \int_0^t h(s) ds.$$

Therefore

$$|Bx(t) - Bx(\tau)| \rightarrow 0 \text{ as } t \rightarrow \tau$$

Again let $\tau \in I_0, t \in I$. Then we obtain

$$|Bx(t) - Bx(\tau)| \leq |\phi(\tau) - \phi(0)| + \left| \int_0^t g(s, x_s) ds \right|$$

$$\leq |\phi(\tau) - \phi(0)| + |p(t) - p(\tau)|$$

Where the function p is defined above. Similarly if $t, \tau \in I_0$, then we get

$$|Bx(t) - Bx(\tau)| \leq |\phi(\tau) - \phi(0)|$$

Therefore in all above three cases

$$|Bx(t) - Bx(\tau)| \rightarrow 0 \text{ as } t \rightarrow \tau, \forall t, \tau \in J$$

Hence $B(C(J, R))$ is an equicontinuous set and consequently $B(C(J, R))$ is relatively compact by Arzela-Ascoli theorem. As a result B is a compact and continuous operator on $C(J, R)$. Thus all the conditions of theorem 2.1 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds.

We show that the conclusion (ii) is not possible. Let $x \in X$ be any solution to FDE(1.1). Then we have, for any $\lambda \in (0, 1)$,

$$\begin{aligned} x(t) &= \lambda A \left(\frac{x}{\lambda} \right)(t) Bx(t) \\ &= \begin{cases} \lambda \left[f \left(t, \frac{x(t)}{\lambda} \right) \left(\phi(0) + \int_0^t g(s, x_s) ds \right) \right], & t \in I \\ \lambda \phi(t), & t \in I_0 \end{cases} \end{aligned}$$

for $t \in J$. Thus if $t \in I_0$, then

$$|x(t)| \leq \lambda \|\phi\|_C.$$

Again if $t \in I$, then we have

$$\begin{aligned} |x(t)| &\leq \lambda \left| f \left(s, \frac{x(t)}{\lambda} \right) \left(\|\phi\|_C + \int_0^t g(s, x_s) ds \right) \right| \\ &\leq \lambda \left(\left| f \left(s, \frac{x(t)}{\lambda} \right) - f(t, 0) \right| + |f(t, 0)| \right) \left(\|\phi\|_C + \int_0^t g(s, x_s) ds \right) \\ &\leq [k(t)|x(t)| + F] \left(\|\phi\|_C + \int_0^t g(s, x_s) ds \right) \\ &\leq k(t) \|x\| \left(\|\phi\|_C + \int_0^t g(s, x_s) ds \right) + F \left(\|\phi\|_C + \int_0^t g(s, x_s) ds \right) \end{aligned}$$

$$\leq \|k\| \|x\| (\|\phi\|_C + \|h\|_{L^1}) + F \int_0^t \gamma(s) \Omega \|x_s\|_C ds. \quad (3.7)$$

Put $u(t) = \sup_{s \in [-r, t]} |x(s)|$, for $t \in J$. Then we have

$$|x(t)| \leq u(t) \quad \forall t \in J \quad \text{and} \quad \|x_t\|_C \leq u(t) \quad \forall t \in I,$$

And so, there is a point $t^* \in [-r, t]$ such that $u(t) = |x(t^*)|$. From (3.7) it follows that

$$u(t) = |x(t^*)|$$

$$\begin{aligned} &\leq \|k\| |x(t^*)| (\|\phi\|_C + \|h\|_{L^1}) + F \left(\|\phi\|_C + \int_0^t \gamma(s) \Omega \|x_s\|_C ds \right) \\ &\leq \|k\| u(t) (\|\phi\|_C + \|h\|_{L^1}) + F \left(\|\phi\|_C + \int_0^t \gamma(s) \Omega (u(s)) ds \right) \\ &= C_1 + C_2 \int_0^t \gamma(s) \Omega (u(s)) ds \end{aligned} \quad (3.8)$$

Where

$$C_1 = \frac{F \|\phi\|_C}{1 - \|k\| (\|\phi\|_C + \|h\|_{L^1})} \quad \text{and} \quad C_2 = \frac{1}{1 - \|k\| (\|\phi\|_C + \|h\|_{L^1})}.$$

Let

$$w(t) = C_1 + C_2 \int_0^t \gamma(s) \Omega (u(s)) ds.$$

Then $u(t) \leq w(t)$ and a direct differentiation of $w(t)$ yields

$$\begin{aligned} w'(t) &\leq C_2 \gamma(t) \Omega (w(t)) \\ w(0) &= C_1, \end{aligned} \quad (3.9)$$

That is

$$\int_0^t \frac{w'(s)}{\Omega(w(s))} ds \leq C_2 \int_0^t \gamma(s) ds \leq C_2 \|\gamma\|_{L^1}$$

A change of variables in the above integral gives that

$$\int_{C_1}^{w(t)} \frac{ds}{\Omega(s)} \leq C_2 \|\gamma\|_{L^1} < \int_{C_1}^{\infty} \frac{ds}{\Omega(s)}.$$

Now an application of mean value theorem yields that there is a constant $M > 0$ such that $w(t) \leq M$ for all $t \in J$. This further implies that

$$|x(t)| \leq \|u(t)\| \leq w(t) \leq M.$$

For all $t \in J$.

Thus the conclusion (ii) of corollary 2.1 does not hold. Therefore the operator Equation $Ax \oplus Bx = x$ and consequently the FDE(1.1) has a solution on J .

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